Homework 3 Solution
PHZ 5156, Computational Physics
September 13, 2005

Problem 1

Terminal velocity is when the acceleration vanishes. That is, $v_x=0$ and $v_z=-v_f$ with (from the condition $a_z=0$) $g=\alpha v_f^2$, or $\alpha=g/v_f^2$.

Completed code:

```
# Homework 3, PHZ 5156, September 13, 2005

from scipy import *

def f_air(yn,tn):
        #RHS of the first-order coupled ODE's
        #Input tn is a particular time.
        #Input yn = array( (x,z,vx,vz) )   at that time.
        #Output:  array( (vx,vz,ax,az) ) at that time.
        x,z,vx,vz = yn
        v = sqrt(vx**2 + vz**2)
        ax = -alpha*v*vx
        az = -alpha*v*vz - g
        f = array( (vx,vz,ax,az) )
        return f

def rk4(f,y0,t):
        #Implements 4th order Runge Kutta, using an external f(y).
        #Usage: y=rk4(f,y0,t)
        #Input f is the name of the "RHS" function.
        #Input y0 is an array that contains the initial conditions.
        #Notice that y0 can have arbitrary length.
        #Input t is an array of times.
        #Output array y has shape len(t) x len(y0) where
        #y[n,:] contains y(tn) for all computed times.

        y = zeros( (len(t),len(y0)), Float)             # Allocate space
        y[0,:] = y0
        for n in arange(0,len(t)-1):
                tn,yn = t[n],y[n,:]
                h1 = tau*f(yn,tn)
                h2 = tau*f(yn+h1/2.,tn+tau/2.)
                h3 = tau*f(yn+h2/2.,tn+tau/2.)
                h4 = tau*f(yn+h3,tn+tau)
                y[n+1,:] = yn + h1/6.+h2/3.+h3/3.+h4/6.
        return y
```

```
# main routine
# Problem 2: Debug f_air
# The following gives v=5 and so ax=-15, az=-30
g = 10.
alpha = 1.
print "Problem 2 test: f_air = ",f_air( array((0.,0.,3.,4.)),5.)
print

# Set up constants
g = 9.8                                 # [m/s^2]
#alpha  = 0.                            # Use this for Problem 4
Vfinal = 120.*1000./3600.
alpha = g / Vfinal**2                   # drag coefficient [1/m] for Prob 5

# Set up initial conditions
v0 = 30000. / 3600.                     # m/s
vx0 = v0 * cos(30.*pi/180.)             # m/s
vz0 = v0 * sin(30.*pi/180.)             # m/s
y0= array( (0.,0.,vx0,vz0) )   # initial conditions

#Times
tau = 1.                                # time step [s]
tfinal = 20.                            # [s]
t = arange(0.,tfinal+tau,tau)  # array of times

#Calculate y at all times, and then unpack into physical variables
y = rk4(f_air,y0,t)

x = y[:,0]
z = y[:,1]
vx = y[:,2]
vz = y[:,3]
print "tau=",tau,"gives final values at t=",t[-1],"of x,z,vx,vz=\n",y[-1,:]

#Analytic solution
xa = vx0*t
za = vz0*t - .5*g*t**2
vxa = vx0*ones(len(t))
vza = vz0 - g*t

#sys.exit()              # uncomment to interrupt code without plotting

#Plotting (written for MacPython)
import Gnuplot,Gnuplot.funcutils
g = Gnuplot.Gnuplot(debug=0)
g('set terminal aqua')
g.title('Trajectory')
g('set data style lines')
```

```
g.xlabel('x [m]')
g.ylabel('z [m]')
d = Gnuplot.Data(x,z,title='numeric',with='linespoints 3 3')
da = Gnuplot.Data(xa,za,title='analytic')
g.plot(d,da)
raw_input('Please press return to continue...\n')

g.reset()
g.title('x as a function of time')
g.xlabel('t [s]')
g.ylabel('x [m]')
g('set data style lines')
d = Gnuplot.Data(t,x,title='numeric',with='linespoints 3 3')
da = Gnuplot.Data(t,xa,title='analytic')
g.plot(d,da)
raw_input('Please press return to continue...\n')

g.reset()
g.title('z as a function of time')
g.xlabel('t [s]')
g.ylabel('z [m]')
g('set data style lines')
d = Gnuplot.Data(t,z,title='numeric',with='linespoints 3 3')
da = Gnuplot.Data(t,za,title='analytic')
g.plot(d,da)
raw_input('Please press return to continue...\n')

g.reset()
g.title('vx as a function of time')
g.xlabel('t [s]')
g.ylabel('vx [m/s]')
g('set data style lines')
d = Gnuplot.Data(t,vx,title='numeric',with='linespoints 3 3')
da = Gnuplot.Data(t,vxa,title='analytic')
g.plot(d,da)
raw_input('Please press return to continue...\n')

g.reset()
g.title('vz as a function of time')
g.xlabel('t [s]')
g.ylabel('vz [m/s]')
g('set data style lines')
d = Gnuplot.Data(t,vz,title='numeric',with='linespoints 3 3')
da = Gnuplot.Data(t,vza,title='analytic')
g.plot(d,da)
```

Problem 2

Executing the code with the test piece

```
# The following gives v=5 and so ax=-15, az=-10-20=-30
g = 10.
alpha = 1.
print "Problem 2 test: f_air = ",f_air( array((0.,0.,3.,4.)),5.)
```

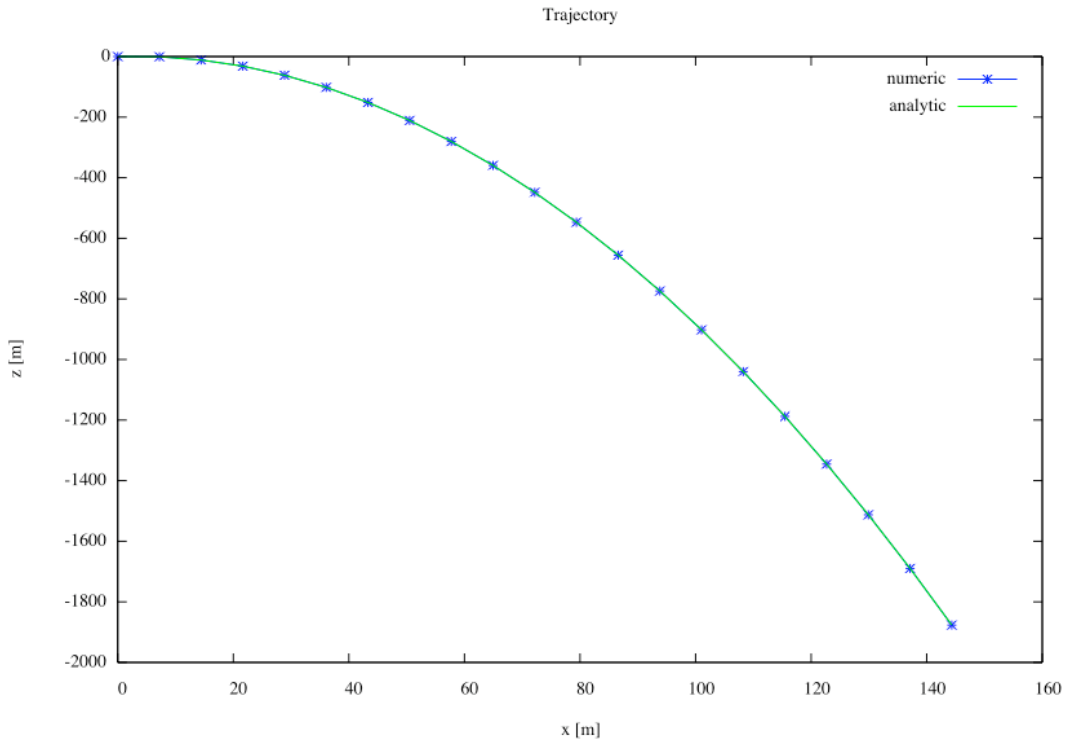
produces the output

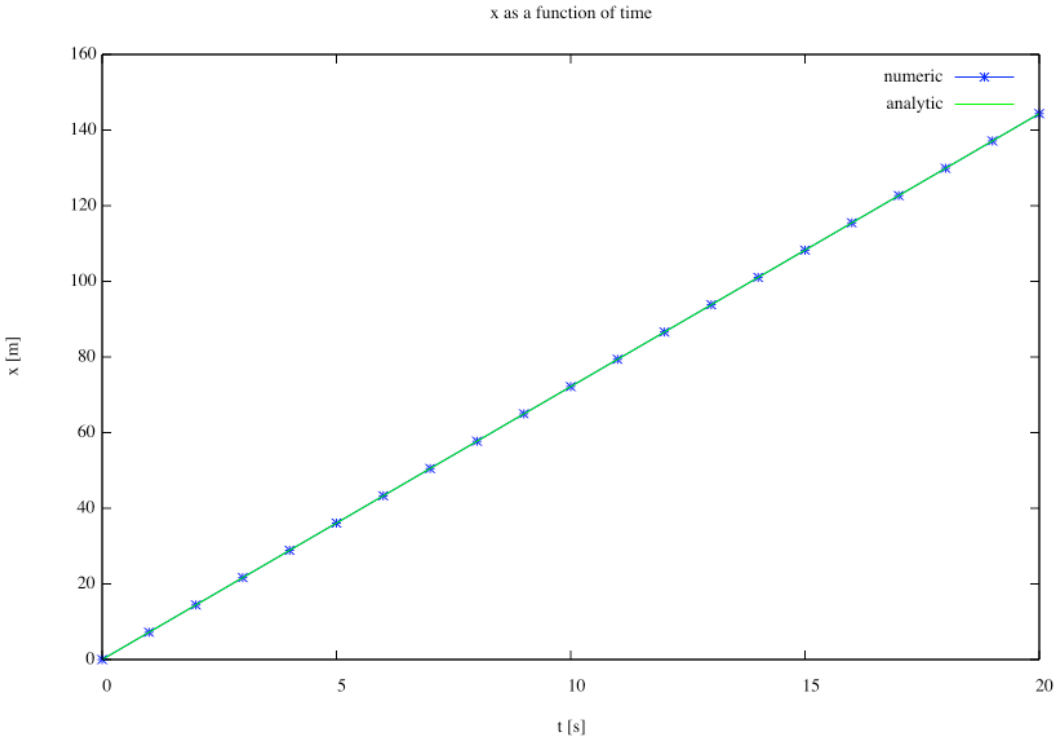
```
Problem 2 test: f_air =  [  3.   4. -15. -30.]
```

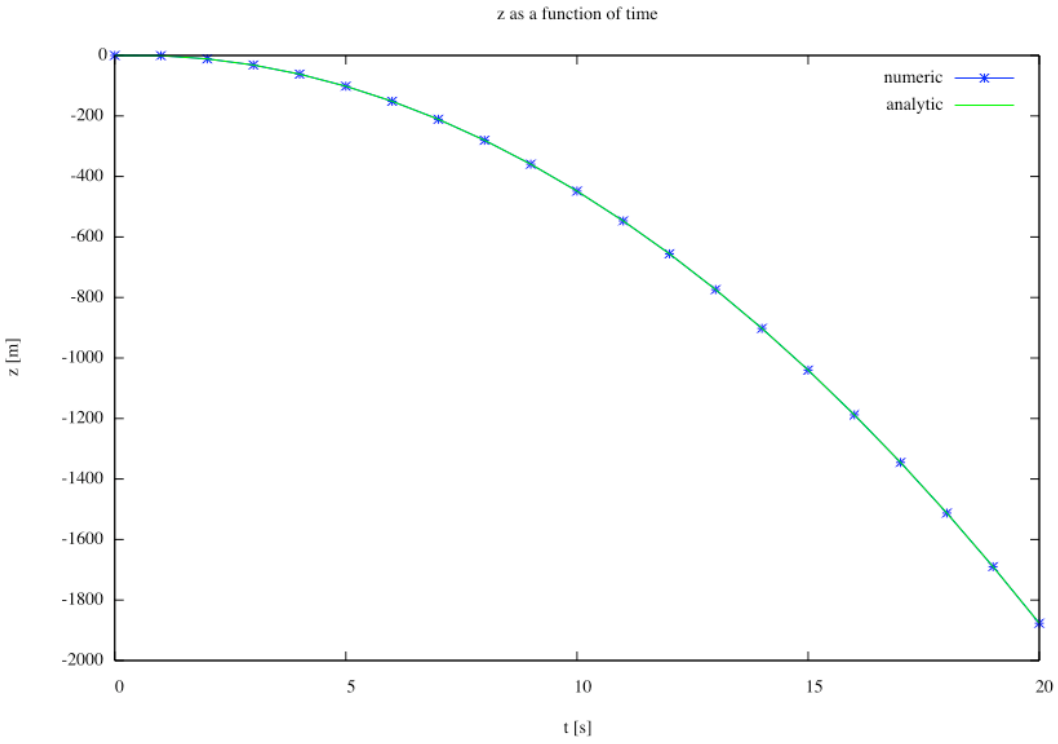
which is clearly correct.

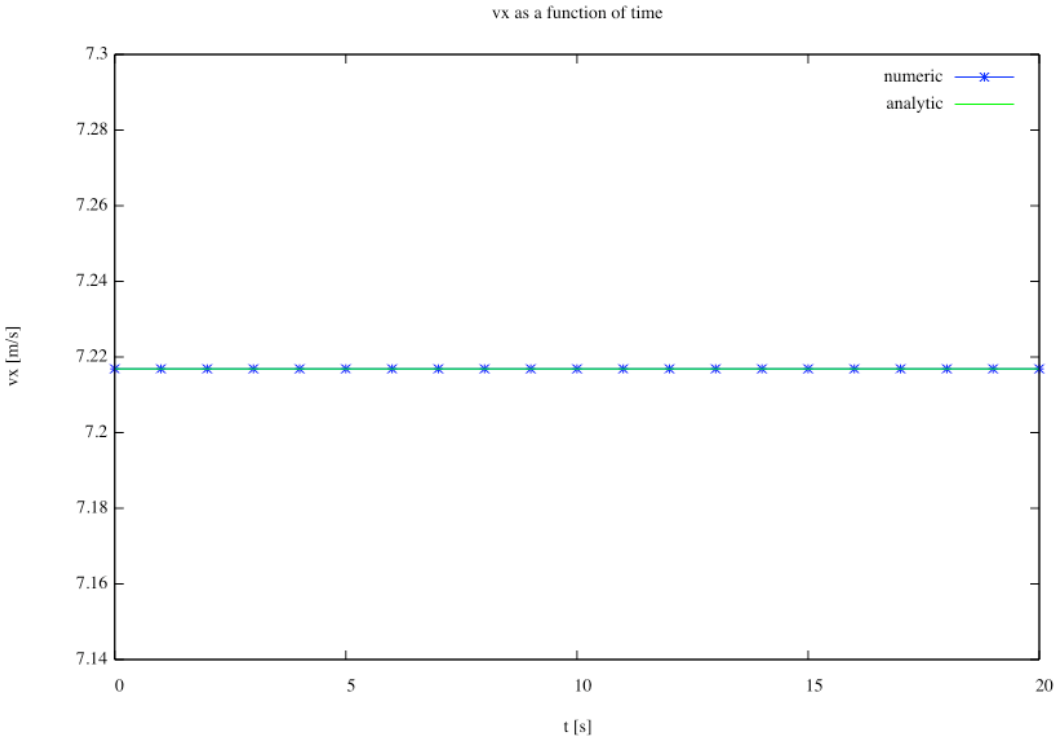
Problem 4

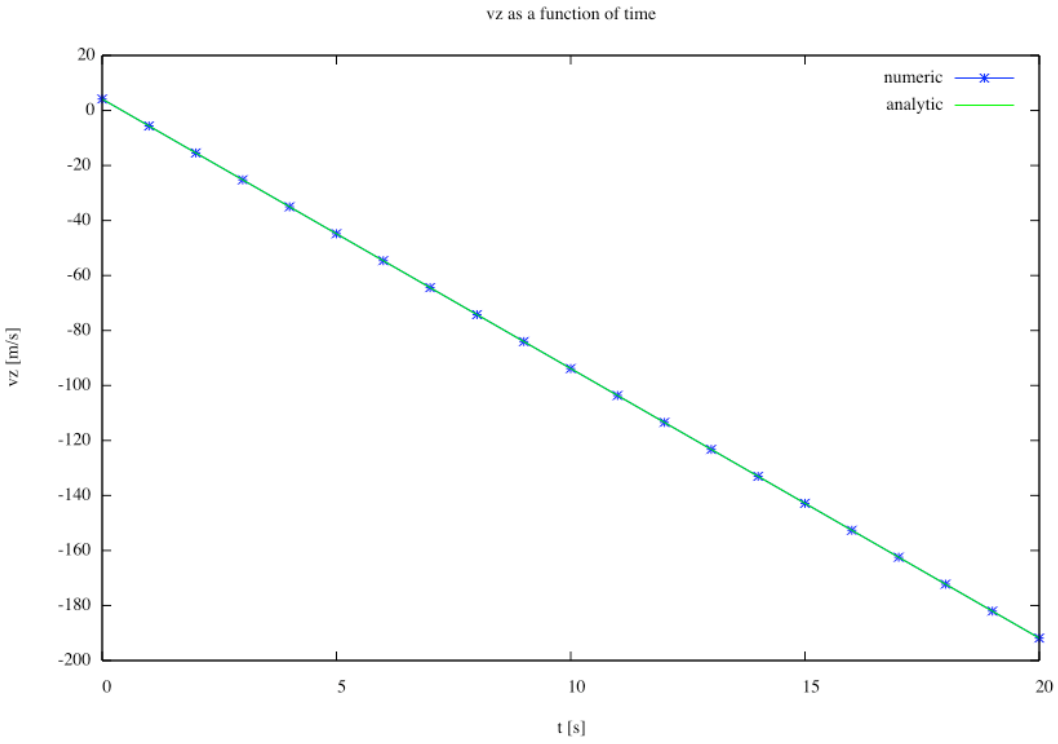
Running with alpha=0 produces the following. The analytic and numerical results agree exactly, a very powerful code check.




x as a function of time
numeric
analytic
x [m]
t [s]
0
20
40
60
80
100
120
140
160
0
5
10
15
20


z as a function of time
numeric
analytic
z [m]
t [s]
0
-200
-400
-600
-800
-1000
-1200
-1400
-1600
-1800
-2000
0
5
10
15
20


vx as a function of time
numeric
analytic
vx [m/s]
t [s]
7.3
7.28
7.26
7.24
7.22
7.2
7.18
7.16
7.14
0
5
10
15
20


vz as a function of time
numeric
analytic
vz [m/s]
t [s]
20
0
-20
-40
-60
-80
-100
-120
-140
-160
-180
-200
0
5
10
15
20

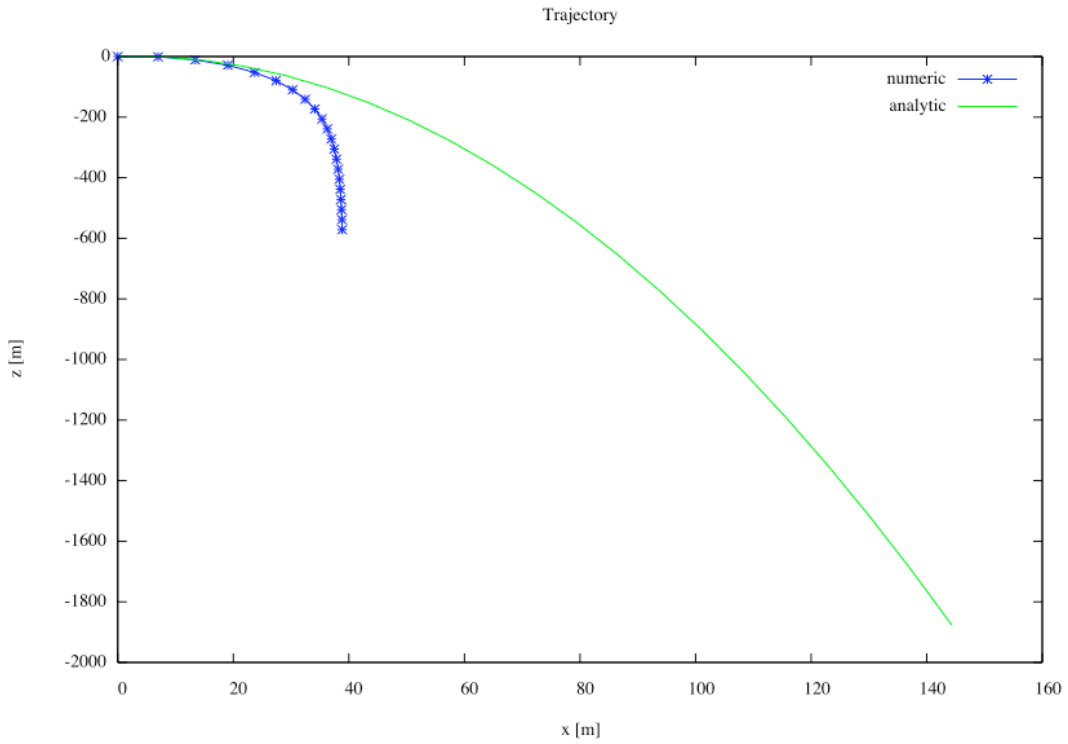
Problem 5

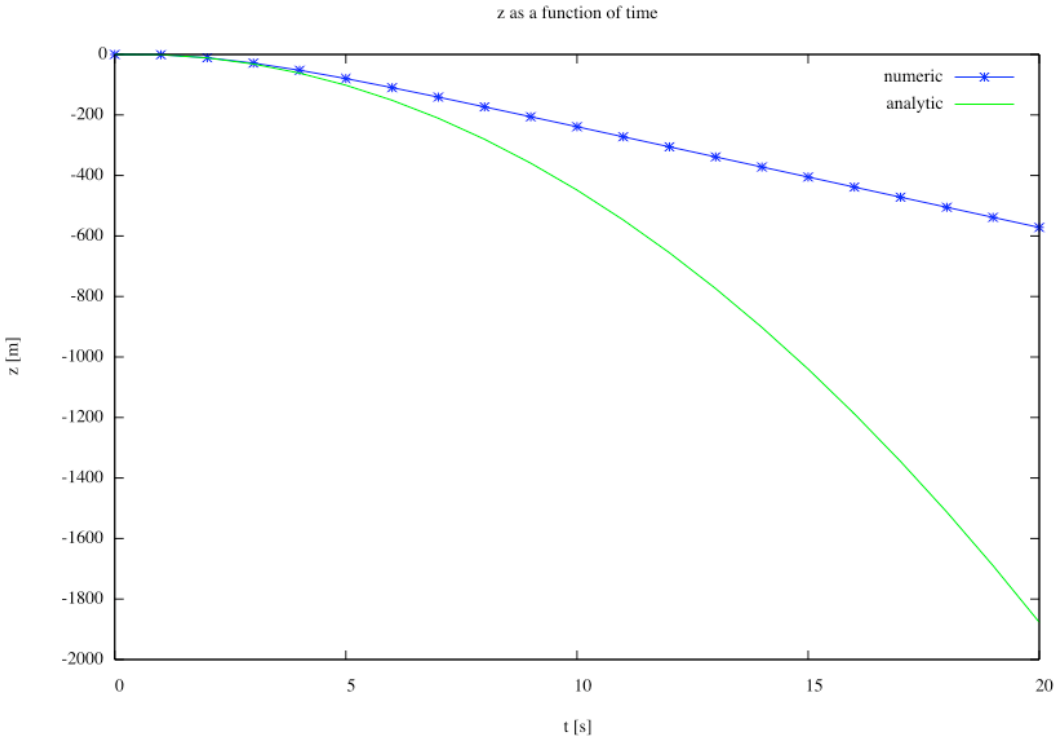
Turning on drag and trying different step sizes tau produces the following final positions and velocities:

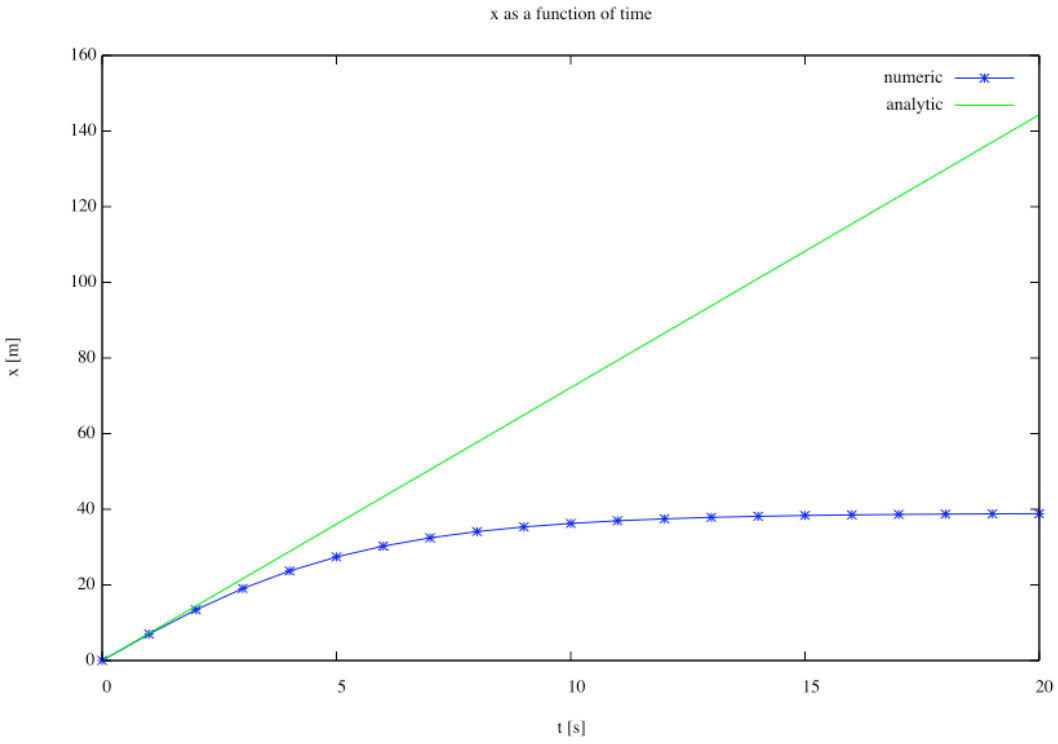
```
tau= 10.0 gives final values at t= 20.0 of x,z,vx,vz=
[  7.60699850e+14  -1.39498536e+16  -5.62403013e+28   1.03134498e+30]

tau= 5.0 gives final values at t= 20.0 of x,z,vx,vz=
[  3.75213065e+01  -5.51641698e+02   2.12144192e-02  -1.96946592e+01]

tau= 1.0 gives final values at t= 20.0 of x,z,vx,vz=
[  3.88401913e+01  -5.71867634e+02   4.23716006e-02  -3.33325041e+01]

tau= 0.1 gives final values at t= 20.0 of x,z,vx,vz=
[  3.88361988e+01  -5.71869968e+02   4.23542258e-02  -3.33325150e+01]

tau= 0.01 gives final values at t= 20.01 of x,z,vx,vz=
[  3.88366216e+01  -5.72203294e+02   4.22298889e-02  -3.33325197e+01]
```

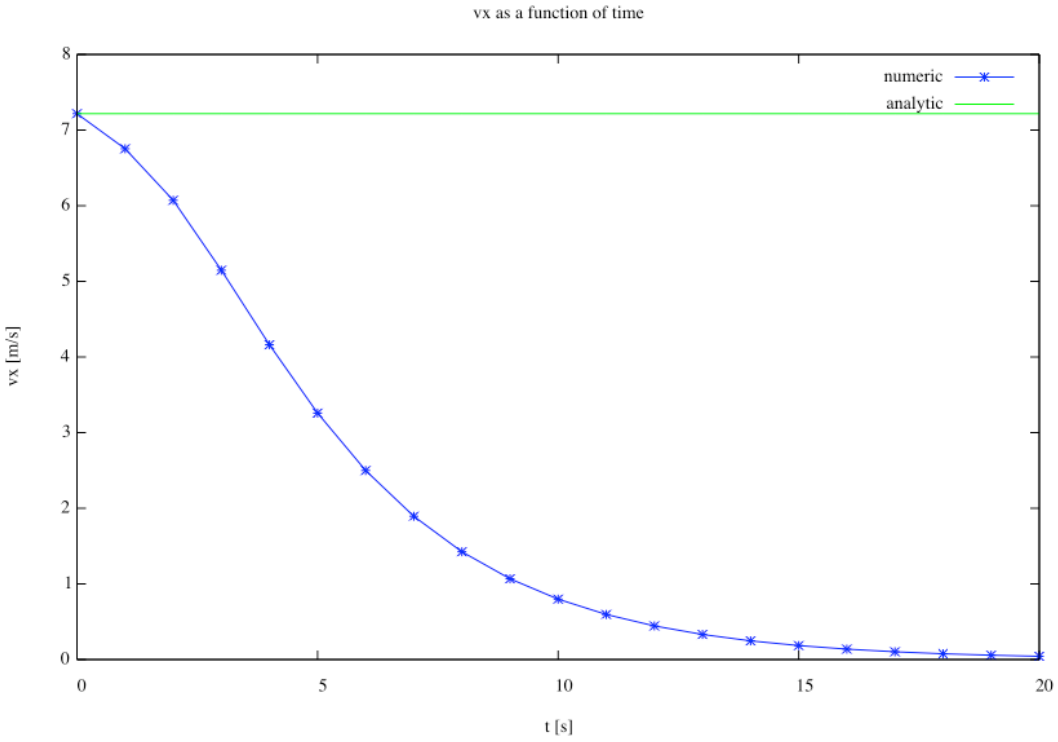
The larger two step sizes tau=10,5 show indications that rk4 may be unstable for this problem if tau is too large. But tau=1 already gives results very close to what smaller tau gives. For quick running and nice plots the remaining plots here are based on tau=1 s.

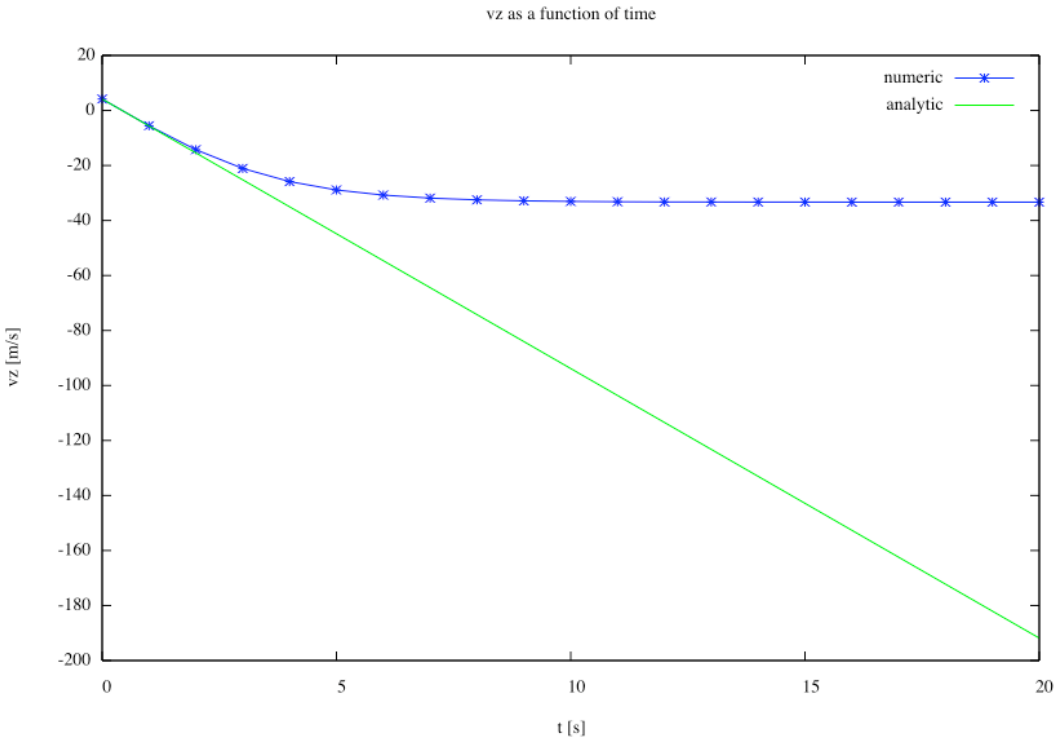
Following are the final plots in the presence of drag.




z as a function of time
numeric
analytic
z [m]
t [s]
0
-200
-400
-600
-800
-1000
-1200
-1400
-1600
-1800
-2000
0
5
10
15
20


x as a function of time
numeric
analytic
x [m]
t [s]
160
140
120
100
80
60
40
20
0
0
5
10
15
20


vx as a function of time
numeric
analytic
vx [m/s]
t [s]
0
1
2
3
4
5
6
7
8
0
5
10
15
20


vz as a function of time
numeric
analytic
vz [m/s]
t [s]
20
0
-20
-40
-60
-80
-100
-120
-140
-160
-180
-200
0
5
10
15
20

Problem 6

Running for 60 seconds produces a final velocity [3.31271192e-07 -3.33333333e+01 m/s. This is nearly exactly the correct terminal velocity of v_x=0 and v_z=-120 km/hr (it is correct to 7 decimal places). This is a very powerful code check.